University of Saskatchewan
Department of Mathematics and Statistics
Math 223 (05, G.Patrick)

October 17, 2005 Test #1 90 minutes

This examination consists of two parts. Part A contains short, routine questions, which you should answer fully but succinctly in the space provided. The questions in Part B are more difficult, and some are designed to challenge you. Fully answer all questions of Part B in the answer books provided.

You should complete Part A rapidly, and save at least half your time to answer the questions in Part B. Part A is worth 30 points and Part B is worth 20 points. Remember to print your name and student ID in the spaces provided in both Part A and Part B.

The points for each problem are indicated in the right margin.

Permitted resources: none.

This is a midterm test. Cheating on an test is considered a serious offense by the University and can be met with disciplinary action, including suspension or expulsion. Candidates shall not bring into the test room any books, resources or papers except at the discretion of the examiner or as indicated on the examination paper. Candidates shall hold no communication of any kind with other candidates within the examination room.

Print your name and student ID here: ______________________

PART A. Fully answer the following questions in the space provided.

Question A1. Determine whether the point $(2,-1,2)$ is in the plane $3x-4y+z-13=0$. 2

$3(2)-4(-1)+2-13 = 6+4+2-13 = 12-13 = -1$

The point does not satisfy the equation, so it is not in the plane

Question A2. Calculate all the unit vectors which are perpendicular to the two vectors $(1,-2,1)$ and $(2,1,3)$. 2

$$\begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = (-6-1, -3+2, 1+4) = (-7,-1,5) \qquad |7,1,5|^2 = 49+1+25 = 75$$

$$\pm \frac{1}{5\sqrt{3}}(-7,-1,5)$$

Question A3. Calculate the projection of the vector $(1,-2,2)$ onto the vector $(0,3,-1)$. 2

$$u=(1,-2,2) \quad v=(0,3,-1) \qquad u\cdot\left(\frac{v}{|v|}\right)\left(\frac{v}{|v|}\right) = \frac{u\cdot v}{|v|^2}v = \frac{-6-2}{9+1}(0,3,-1)$$

$$= -\frac{4}{5}(0,3,-1)$$

Question A4. Calculate the unit tangent vector, in the direction of increasing t, to the curve $\mathbf{r}(t) = (t^2, 1-t-2t^2, t)$ at $t=1$. 2

$r'(t) = (2t, -1-4t, 1) \qquad r'(1) = (2, -5, 1) \qquad |r'(1)|^2 = 4+25+1 = 30$

$\hat{T} = \frac{1}{\sqrt{30}} (2, -5, 1)$

Question A5. If $u(t)$ and $v(t)$ are two curves such that 2

$$u(0) = (2,-1,0), \quad v(0) = (1,1,1), \quad u'(0) = (1,-1,1), \quad v'(0) = (-2,1,2)$$

then calculate the value of $(u \cdot v)'(0)$.

$(u \cdot v)'(0) = u'(0) \cdot v(0) + u(0) \cdot v'(0)$

$= (1,-1,1) \cdot (1,1,1) + (2,-1,0) \cdot (-2,1,2)$

$= (1-1+1) + (-4-1+0) = -4$

Question A6. With exactly the same information as in the previous question, calculate the value of $(u \times v)'(0)$. 2

$(u \times v)'(0) = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} i & j & k \\ 2 & -1 & 0 \\ -2 & 1 & 2 \end{vmatrix}$

$= (-2, 0, 2) + (-2, -4, 0) = (-4, -4, 2)$

Question A7. If a particle follows a path in such a way that the curvature of the path at time $t=1$ is $\kappa = \frac{1}{2}$ and the speed of the particle at time $t=1$ is 4, then calculate the normal component of the particle's acceleration. 2

$a_N = \frac{v^2}{r} = \kappa v^2 = \frac{1}{2} \cdot 16 = 8$

Question A8. Suppose that a particle of mass 2 is acted upon by a force $(1-t, t, t^2)$, and that the particle is at rest at the point $(1, -2, 1)$ when $t = 0$. Calculate the position and speed of the particle at $t = 2$. 2

$$a = \left(\tfrac{1}{2}-\tfrac{1}{2}t, \tfrac{1}{2}t, \tfrac{1}{2}t^2\right) \qquad v = (0,0,0) + \int_0^t \left(\tfrac{1}{2}-\tfrac{1}{2}t, \tfrac{1}{2}t, \tfrac{1}{2}t^2\right) dt$$

$$v(2) = \left(0, 1, \tfrac{8}{6}\right)$$

$$= \left(\frac{t}{2} - \frac{1}{4}t^2, \frac{1}{4}t^2, \frac{1}{6}t^3\right) \qquad |v| = \frac{5}{3}$$

$$r = (1,-2,1) + \int_0^2 \left(\frac{t}{2} - \frac{1}{4}t^2, \frac{1}{4}t^2, \frac{1}{6}t^3\right) = (1,-2,1) + \left(\frac{t^2}{4} - \frac{t^3}{12}, \frac{t^3}{12}, \frac{t^4}{24}\right)\Big|_{t=0}^{t=2}$$

$$= (1,-2,1) + \left(1-\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) = \left(\frac{4}{3}, \frac{-4}{3}, \frac{5}{3}\right)$$

Question A9. Prove that the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist. 2

$$y = mx \qquad \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2},$$

which depends on m.

Question A10. Calculate the gradient of the function $f(x, y, z) = \sin(xy^2) + zy$ at $(\pi, 2, -1)$. 2

$$\nabla f(\pi,2,-1) = \left(y^2\cos(xy^2), 2xy\cos(xy^2)+z, y\right)(\pi,2,-1)$$

$$= \left(4\cdot\cos 4\pi, 4\pi\cos(4\pi)-1, 2\right)$$

$$= (4, 4\pi-1, 2)$$

Question A11. Find all the critcial points of the function $f(x, y) = 2x^3 + 9x^2 + y^2 + 12x - 2y$. 2

$$\frac{\partial f}{\partial x} = 6x^2 + 18x + 12 = 6(x^2+3x+2) = 0 \iff x = -2,\ x = -1$$

$$\frac{\partial f}{\partial y} = 2y - 2 = 0 \iff y = 1$$

Critical points are $(-1, 1)$, $(-2, 1)$

Question A12. Verify that the function $z = y^2+2xy+6x-2y$ has a critical point at $(x, y) = (4, -3)$). Is this critical point a local maximum, a local minimum, or a saddle point? 2

$$\frac{\partial z}{\partial x} = 2y+6 = 0 \quad \text{at } y=-3 \qquad \frac{\partial z}{\partial y} = 2y+2x-2 = -6+8-2=0$$

$$\text{at } (x,y)=(4,-3)$$

$$A = \frac{\partial^2 z}{\partial x^2} = 0 \quad B = \frac{\partial^2 z}{\partial x \partial y} = 2 \quad C = \frac{\partial^2 z}{\partial y^2} = 2$$

$$B^2 - AC = 4 - 0\cdot 2 = 4 > 0 \Rightarrow \text{saddle.}$$

Question A13. Compute the equation of the tangent plane to $z = \ln(x+2y)+xy$ at $(x, y) = (-1, 1)$. 2

$$f = z - \ln(x+2y) - xy \; ; \quad \text{at } x=-1, y=1, \; z = \ln(-1+2) - 1 = -1$$

$$\nabla f(-1,1,-1) = \left(-\frac{1}{x+2y} - y, \; \frac{-2}{x+2y} - x, \; 1 \right) = (-1-1, -2+1, 1) = (-2,-1,1)$$

$$(-2,-1,1)\cdot(x+1, y-1, z+1) = -2x - y + z = 0$$

$$-2x - y + z = 0$$

Question A14. If $z = \ln(x^2+y^3)$ then compute $\dfrac{\partial^2 z}{\partial x \partial y}$ and $\dfrac{\partial^2 z}{\partial x^2}$ at $(x, y) = (1, 1)$. 2

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2+y^3} \qquad \frac{\partial^2 z}{\partial x^2}(1,1) = \left. \frac{2(x^2+y^3) - 2x(2x)}{(x^2+y^3)^2} \right|_{\substack{x=1 \\ y=1}} = \frac{2\cdot 2 - 2\cdot 2}{2^2} = 0$$

$$\frac{\partial^2 z}{\partial x \partial y}(1,1) = \frac{-6y^2 x}{(x^2+y^3)^2} = \frac{-6}{4} = -\frac{3}{2}$$

Question A15. If $w = s^2+t^3+2r^2$, $s = xy+z$, $t = x^2+3yz$, $r = x^2+2z^2$, then use the chain rule to show that $\dfrac{\partial w}{\partial z} = 2s + 9t^2y + 16rz$. 2

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial z} + \frac{\partial w}{\partial r}\frac{\partial r}{\partial z}$$

$$= 2s\cdot 1 + 3t^2\cdot 3y + 4r\cdot 4z$$

$$= 2s + 9yt^2 + 16rz$$

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Solutions

Verify: $1^2 - 1\cdot\cos\left(\frac{\pi}{2}\cdot 0\right) + 0^2 = 1^2 - 1 = 0$

$1^2 + 1^2 - \sin\left(\frac{\pi}{2}\cdot 0\right) + 2\cdot 0^2 = 2$

$1\cdot 1 - \sin\frac{\pi}{2}\cos 0 + 0 = 1 - 1 = 0$

$2x x_u - y_u \cos uv + y v \sin uv + 2z z_u = 0 \qquad 2x_u - y_u = 0 \qquad ①$

$2x x_u + 2y y_u - v\cos(uv) + 4z z_u = 0 \qquad 2x_u + 2y_u = 0 \qquad ②$

$x_u y + x y_u - \cos u \cos v + z_u = 0 \qquad x_u + y_u + z_u = 0 \qquad ③$

② − ① : $3y_u = 0 \quad y_u = 0 \quad x_u = \frac{1}{2} y_u = 0, \quad z_u = -x_u - y_u = 0$

$$\frac{\partial x}{\partial u} = 0, \quad \frac{\partial y}{\partial u} = 0, \quad \frac{\partial z}{\partial u} = 0$$

Q2.

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial x} = \frac{\partial f}{\partial s} + 2x\frac{\partial f}{\partial t}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial y} = \frac{\partial f}{\partial s} + 2y\frac{\partial f}{\partial t}$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial z} = 2z\frac{\partial f}{\partial t}$$

$$z\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) - (x+y)\frac{\partial u}{\partial z} = z(2x+2y)\frac{\partial f}{\partial t} - (x+y)\cdot 2z\frac{\partial f}{\partial t} = 0$$

B3 $\nabla(x^2y+y^2z) = (2xy,\ x^2+2yz,\ y^2)$, so the normal line is along $(-2,-1,1)$. The vector equation of that line is

$$(x,y,z) = (1,-1,1)+t(-2,-1,1) = (1-2t,\ -1-t,\ 1+t)$$

Put this (x,y,z) into the eq of the plane:

$$(1-2t)+2(-1-t)-(1+t) = -2-5t = 4$$

$$5t = -6 \quad t = -\frac{6}{5}$$

So the intersection is at $\left(1+\frac{12}{5},\ -1+\frac{6}{5},\ 1-\frac{6}{5}\right) = \left(\frac{17}{5}, \frac{1}{5}, -\frac{1}{5}\right)$

B4 $r(t) = (e^t\cos t,\ e^t\sin t,\ e^t)$

$$|r'(t)|^2 = (e^t\cos t - e^t\sin t)^2 + (e^t\sin t + e^t\cos t)^2 + (e^t)^2$$

$$= 3e^{2t}$$

$$s = \int_0^t \sqrt{3}e^t\,dt = \sqrt{3}(e^t-1) \qquad t = \ln\left(\frac{s}{\sqrt{3}}+1\right)$$

$$r(s) = \left(\frac{s}{\sqrt{3}}+1\right)\left(\cos\ln\left(\frac{s}{\sqrt{3}}+1\right),\ \sin\ln\left(\frac{s}{\sqrt{3}}+1\right),\ 1\right), \quad s \in (0,\infty)$$